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The motion of long slender bodies in a viscous fluid
Part 1. General theory

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A solid long slender body is considered placed in a fluid undergoing a given undisturbed flow. Under conditions in which fluid inertia is negligible, the force per unit length on the body is obtained as an asymptotic expansion in terms of the ratio of the cross-sectional radius to body length. Specific examples are given for the resistance to translation of long slender bodies for cases in which the body centre-line is curved as well as for those for which the centre-line is straight.

1. Introduction

There are only relatively few problems in which it is possible to solve exactly the creeping motion equations for flow around a single isolated solid body. Thus Stokes (1851) calculated the flow around a solid sphere undergoing uniform translation through a viscous fluid whilst Oberbeck (1876) considered the analogous problem for the spheroid and Payne & Pell (1960) obtained general solutions for the case of axisymmetric flow relative to lens-shaped bodies. Brenner (1964) obtained the flow around a solid body whose shape was that of a slightly deformed sphere, the velocity field being obtained as an expansion in terms of the deformation.

Since many particles encountered in practice are of irregular shape, it is of interest to investigate the flow around a class of bodies of irregular shape for which one may solve the creeping motion equations. In this paper, we therefore consider the flow around a long slender solid body which may or may not be straight. If such a body is of length l and has a cross-sectional radius of order b, an expansion of the velocity field about such a body is made in terms of the parameter $\kappa = b/l$. Such a body is a suitable model for a fibre or thread-like particle which may be either rigid or flexible. The behaviour of such a kind of flexible fibre in shear flow was investigated experimentally by Forgacs & Mason (1959*a*, *b*).

The general theory of long slender bodies described in this paper will be used to find the hydrodynamic forces exerted on such a particle when it is placed in a general undisturbed flow field. It is also shown how the theory may be modified to investigate (i) the mutual interaction between two or more long slender bodies and (ii) the behaviour of a long slender body in the neighbourhood of solid walls.

In the final section (§ 8), there are brief descriptions of how the general theory may be employed in a number of particular problems of interest.

2. Translational resistance of long slender bodies

Several results pertaining to the uniform translational motion of long slender bodies are now discussed. Consider a body with surface S at rest in a fluid which at infinity is undergoing a uniform translational motion with velocity $\mathbf{U}$. Then if fluid inertia effects are neglected, the velocity $\mathbf{v}$ and pressure p in the fluid satisfy the creeping motion equations

$$\mu\nabla^2\mathbf{u} - \nabla p = 0, \quad \nabla . \mathbf{u} = 0, \tag{2.1}$$

with the boundary conditions

$$\begin{aligned} \mathbf{u} &= 0 \quad \text{on} \quad S, \\ \mathbf{u} &\to \mathbf{U} \quad \text{as} \quad \mathbf{r} \to \infty, \end{aligned} \tag{2.2}$$

μ being the fluid viscosity and $\mathbf{r}$ the position vector of a general point. Oberbeck (1876) solved this problem in the particular case of an ellipsoid and obtained the force $\mathbf{F}$ exerted on such a body. In particular for a spheroid with semi-axes a and b with a measured along the symmetry axis, Oberbeck's formula for $\mathbf{U}$ in the direction of the symmetry axis leads to a value for the force $\mathbf{F}$ (also along the symmetry axis) of magnitude

$$F = 16\pi\mu b U\left[-\frac{2(a/b)}{(a/b)^2-1} + \frac{2(a/b)^2-1}{\{(a/b)^2-1\}^{\frac{3}{2}}} \ln\left(\frac{a/b+((a/b)^2-1)^{\frac{1}{2}}}{a/b-((a/b)^2-1)^{\frac{1}{2}}}\right)\right]^{-1}, \tag{2.3}$$

which for large axis ratios (i.e. for small b/a), reduces to

$$F = 4\pi\mu a U\left\{\frac{1}{\ln(2a/b)-0{\cdot}5} + O(b/a)\right\} \quad \text{as} \quad (b/a) \to 0. \tag{2.4}$$

Burgers (1938) attempted to obtain the formula (2.4) directly for a long slender ellipsoid of revolution. He assumed that the disturbance produced by the ellipsoid was like that which would be produced by a line of force of magnitude

$$\left.\begin{aligned} f(z) &= A_0 + A_2(z/a)^2 + A_4(z/a)^4 \quad \text{if} \quad |z| < a, \\ f(z) &= 0 \quad \text{otherwise}, \end{aligned}\right\} \tag{2.5}$$

acting along the symmetry axis, z being distance measured along the symmetry axis from the centre of the ellipsoid and A_0, A_2, A_4 being constants. Burgers then evaluated the complete velocity field $\mathbf{u}$ consisting of the uniform flow $\mathbf{U}$ and the disturbance produced by the line of force and then found the values of A_0, A_2 and A_4 which minimized the mean value of $|\mathbf{u}|$ on the surface of the body (upon which one would like to satisfy $\mathbf{u} = 0$). This procedure yielded the value of the force on the ellipsoid as

$$F = \frac{4\pi\mu a U}{\ln(2a/b) - 0{\cdot}5}, \tag{2.6}$$

which is exactly the same as the equation (2.4) obtained from Oberbeck's results. Burgers also used his method to determine the force on a circular cylinder of finite length which was held at rest in a uniform stream flowing in the direction of the symmetry axis. For this case, he obtained the value of the force as

$$F = \frac{4\pi\mu a U}{\ln(2a/b) - 0{\cdot}72}, \tag{2.7}$$

where a is the semi-length and b the radius of the cylinder.

Broersma (1960) improved the method used by Burgers by taking the disturbance produced by the body as being that produced by a line of force of magnitude

$$\left.\begin{aligned} f(z) &= B_0+B_2(z/a)^2+B_4(z/a)^4+\ldots \quad \text{if} \quad |z| < a, \\ &= 0 \qquad\qquad\qquad\qquad\qquad\qquad\quad \text{otherwise}, \end{aligned}\right\} \tag{2.8}$$

where B_0, B_2, B_4, ... are an infinite set of constants to be determined. If one now evaluates $\mathbf{u}$ and minimizes the mean value of $|\mathbf{u}|$ on the surface of the body one obtains an infinite set of linear equations for B_0, B_2, B_4, ... Broersma computed the values of these constants numerically for the case of a circular cylinder of finite length with $\mathbf{U}$ in the direction of the symmetry axis and obtained for the force on the cylinder

$$F = \frac{4\pi\mu a U}{\ln(2a/b)-0{\cdot}81}, \tag{2.9}$$

which differs from the result (2.7) obtained by Burgers. It may therefore be concluded that the method used by Burgers is not asymptotically correct in the limit of $b/a \to 0$ to the order considered. However, it is not obvious whether Broersma's method gives an asymptotically correct solution to the order given in (2.9) and if correct, what the next term in the expansion might be.

In the present paper, this point is clarified since we shall find the force on a long slender body as an asymptotic expansion in terms of the ratio of body cross-section to body length.

Tuck (1964) has investigated the translational resistance force on slender bodies by the use of spheroidal co-ordinates while recently Tillett (1970) and Batchelor (1970) have used a method similar to that described in this paper to find respectively the hydrodynamic resistance of slender bodies of revolution and straight slender bodies of non-circular cross-section.

3. General problem

Consider a long slender body S of circular cross-section, the length of the body being l and a characteristic value of the cross-sectional radius being b. This body may be assumed bent in any manner whatsoever so long as the radius of curvature of such a bending is at all points of order l. The distance along the body centre-line measured from one end is s' (see figure 1) and a dimensionless quantity s is given by

$$s = s'/l \tag{3.1}$$

so that $0 \leqslant s \leqslant 1$, the two ends of the body S being $s = 0$ and $s = 1$. The cross-sectional radius at any point of the centre-line is taken to be $b\lambda(s)$, where $\lambda(s)$ is a dimensionless function of s.

Dimensionless quantities will be used (unless otherwise stated) based upon the length l, the fluid viscosity μ and a characteristic velocity U. The vector $\mathbf{r}$ is now defined as the dimensionless position of a general point relative to a fixed set of rectangular Cartesian co-ordinates with origin O. The body S is considered placed in an undisturbed flow field with dimensionless value $\mathbf{U}(\mathbf{r})$, this flow field itself satisfying the creeping motion equations, i.e.

$$\nabla^2\mathbf{U}-\nabla P = 0, \quad \nabla . \mathbf{U} = 0, \tag{3.2}$$

P being a dimensionless pressure field. It is assumed also that the velocity of material points on the centre-line of the body is given by a function $\mathbf{U}^*(s)$ and that the centre-line itself is given by $\mathbf{r} = \mathbf{R}(s)$.

The complete velocity field (i.e. the flow field $\mathbf{U}$ together with the disturbance flow produced by S) is defined as $\mathbf{u}$, this flow field also satisfying the creeping motion equations

$$\nabla^2\mathbf{u} - \nabla p = 0, \quad \nabla . \mathbf{u} = 0, \tag{3.3}$$

p being the pressure field corresponding to $\mathbf{u}$.

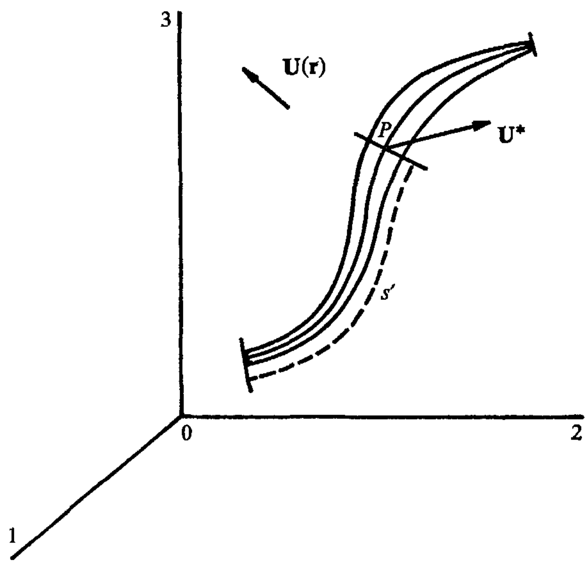

FIGURE 1. Long slender body moving in a fluid undergoing an undisturbed flow $\mathbf{U}(\mathbf{r})$.

The parameter κ is defined by

$$\kappa = b/l \tag{3.4}$$

and is assumed to be very small compared with unity. It is in terms of this parameter that we shall make expansions of the velocity field. However, one should note that this type of expansion must be singular since the flow locally around the long thin body must be very nearly a flow around an infinite circular cylinder and it is well known (Stokes paradox) that it is impossible for a flow field $(\mathbf{u}, p)$ to satisfy the creeping motion equations and at the same time to satisfy the no slip condition $\mathbf{u} = 0$ on an *infinite* circular cylinder and also to make $\mathbf{u}$ tend to a uniform flow at infinity.

Using dimensionless variables, one therefore defines an *outer* expansion in κ for which $\mathbf{r}$ is used as the independent variable and $\mathbf{u}$ and p as dependent variables. At each point P (at $\mathbf{r} = \mathbf{R}_P$) of the centre-line of the body S one may define an *inner* expansion in κ for which $\bar{\mathbf{r}}$ is used as the independent variable and $\bar{\mathbf{u}}$ and $\bar{p}$ as dependent variables where $\bar{\mathbf{r}}$, $\bar{\mathbf{u}}$ and $\bar{p}$ are given by

$$\bar{\mathbf{r}} = (\mathbf{r} - \mathbf{R}_P)/\kappa, \quad \bar{\mathbf{u}} = \mathbf{u}, \quad \bar{p} = \kappa p. \tag{3.5}$$

In the outer expansion, l is the unit of length and as $\kappa \to 0$, the body S becomes a line singularity (i.e. $b \to 0$), whereas in the inner expansion at each point P of the centre-line, the unit of length is b and as $\kappa \to 0$, the body S becomes very much like a cylinder of infinite length (since $l \to \infty$). Actually one has an infinite number of inner expansions corresponding to each point of the centre-line of the body S. However, all such inner expansions may be considered simultaneously by taking a general point P of the body centre-line. The inner expansion at such a point is then matched onto the solution for the outer expansion at the same point P.

4. Inner expansion

At a general point P of the centre-line of the body S, consider the inner expansion. By expressing the creeping motion equations (3.3) in terms of the inner variables, one obtains

$$\bar{\nabla}^2\bar{\mathbf{u}} - \bar{\nabla}\bar{p} = 0, \quad \bar{\nabla}.\bar{\mathbf{u}} = 0, \tag{4.1}$$

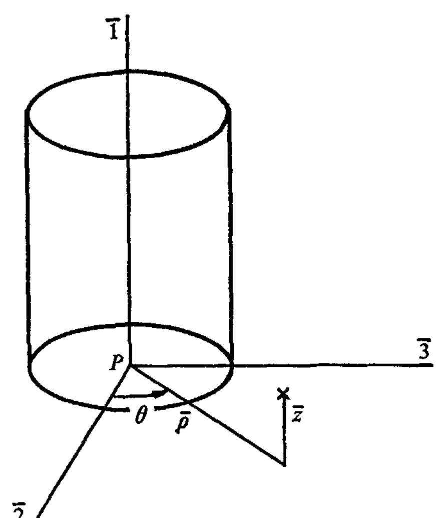

FIGURE 2. Cylindrical co-ordinate system $(\bar{\rho}, \theta, \bar{z})$.

where all derivatives are with respect to the $\bar{\mathbf{r}}$ variables. These equations are to be solved with the no slip boundary condition that

$$\bar{\mathbf{u}} = \mathbf{U}^*(s) \text{ on the surface of the body.} \tag{4.2}$$

Locally a set of rectangular Cartesian axes $(\bar{r}_1, \bar{r}_2, \bar{r}_3)$ are taken in such a manner that the $\bar{r}_1$ axis lies in the direction of the tangent to the centre-line at P and relative to these axes, a cylindrical polar system of axes $(\bar{\rho}, \theta, \bar{z})$ is considered (see figure 2) such that

$$\bar{r}_1 = \bar{z}, \quad \bar{r}_2 = \bar{\rho}\cos\theta, \quad \bar{r}_3 = \bar{\rho}\sin\theta. \tag{4.3}$$

If s_P is the value of s at the point P, the surface of the body S in the neighbourhood of P may be expressed in the form

$$\bar{\rho} = \lambda(s_P) + O(s - s_P),$$

$\lambda(s)$ being defined in § 3. Now since $(s-s_P)$ is approximately equal to $\kappa\bar{z}$ it follows that the surface of the body S may be written as

$$\bar{\rho} = \lambda(s)+O(\kappa).$$

The inner boundary condition on $\bar{\mathbf{u}}$ is therefore

$$\bar{\mathbf{u}} = \mathbf{U}^*(s) \quad \text{on} \quad \bar{\rho} = \lambda(s)+O(\kappa). \tag{4.4}$$

If one writes

$$\bar{\mathbf{u}} = \bar{\mathbf{u}}_0+O(\kappa), \quad \bar{p} = \bar{p}_0+O(\kappa), \tag{4.5}$$

then $(\bar{\mathbf{u}}_0, \bar{p}_0)$ must satisfy

$$\bar{\nabla}^2\bar{\mathbf{u}}_0-\bar{\nabla}\bar{p}_0 = 0, \quad \bar{\nabla}.\bar{\mathbf{u}}_0 = 0, \tag{4.6}$$

with

$$\bar{\mathbf{u}}_0 = \mathbf{U}^*(s) \quad \text{on} \quad \bar{\rho} = \lambda(s). \tag{4.7}$$

No outer boundary condition will be imposed on $\bar{\mathbf{u}}_0$ at this stage since such a boundary condition is determined by the required matching onto the outer expansion. The general solution of equations (4.6) with boundary conditions (4.7) may be written as

$$\left.\begin{aligned}
(\bar{u}_0)_{\bar{\rho}} &= C(\kappa)\{1-\lambda^2\bar{\rho}^{-2}-2\ln(\bar{\rho}/\lambda)\}\cos\theta+(U^*)_{\bar{2}}\cos\theta \\
&\quad +D(\kappa)\{1-\lambda^2\bar{\rho}^{-2}-2\ln(\bar{\rho}/\lambda)\}\sin\theta+(U^*)_{\bar{3}}\sin\theta, \\
(\bar{u}_0)_{\theta} &= C(\kappa)\{1-\lambda^2\bar{\rho}^{-2}+2\ln(\bar{\rho}/\lambda)\}\sin\theta-(U^*)_{\bar{2}}\sin\theta \\
&\quad +D(\kappa)\{-1+\lambda^2\bar{\rho}^{-2}-2\ln(\bar{\rho}/\lambda)\}\cos\theta+(U^*)_{\bar{3}}\cos\theta, \\
(\bar{u}_0)_{\bar{z}} &= E(\kappa)\ln(\bar{\rho}/\lambda)+(U^*)_{\bar{1}}, \\
\bar{p}_0 &= C(\kappa)\{4\bar{\rho}^{-1}\cos\theta\}+D(\kappa)\{4\bar{\rho}^{-1}\sin\theta\}+F(\kappa),
\end{aligned}\right\} \tag{4.8}$$

where $C(\kappa)$, $D(\kappa)$, $E(\kappa)$ and $F(\kappa)$ are arbitrary constants which may be taken as functions of κ. The expression in (4.8) for $(\bar{u}_0, \bar{p}_0)$ should also contain terms in $\cos n\theta$ and $\sin n\theta$ with $n \geqslant 2$. However these have been omitted since it may be shown that if they are included their coefficients must be zero to the order in κ which is considered (otherwise they would give rise to terms like κ^{-n} in the outer expansion with $n > 0$). For the quantities $C(\kappa)$, $D(\kappa)$, $E(\kappa)$ and $F(\kappa)$ one must choose a suitable expansion in terms of the parameter κ. A careful examination of equations (4.8) leads one to the conclusion that these expansions for $C(\kappa)$, $D(\kappa)$, $E(\kappa)$ and $F(\kappa)$ must be of the form

$$\left.\begin{aligned}
C(\kappa) &= \frac{C_1}{\ln\kappa}+\frac{C_2}{(\ln\kappa)^2}+\ldots, \\
D(\kappa) &= \frac{D_1}{\ln\kappa}+\frac{D_2}{(\ln\kappa)^2}+\ldots, \\
E(\kappa) &= \frac{E_1}{\ln\kappa}+\frac{E_2}{(\ln\kappa)^2}+\ldots, \\
F(\kappa) &= \kappa F_0+\frac{\kappa F_1}{\ln\kappa}.
\end{aligned}\right\} \tag{4.9}$$

The substitution of $C(\kappa)$, $D(\kappa)$ and $E(\kappa)$ from (4.9) into the expressions (4.8) for $(\bar{\mathbf{u}}_0, \bar{p}_0)$ and the changing of the resulting equations to outer variables yields the inner boundary condition on the flow field $(\mathbf{u}, p)$ of the *outer* expansion. Using (ρ, θ, z) polar axes, corresponding to the $(\bar{\rho}, \theta, \bar{z})$ axes so that

$$\rho = \kappa\bar{\rho} \quad \text{and} \quad z = \kappa\bar{z}, \tag{4.10}$$

the inner boundary conditions on the *outer* flow field $(\mathbf{u}, p)$ may be written as

$$\left.\begin{aligned}
u_\rho &\sim \{(2C_1 + U_{\bar{2}}^*)\cos\theta + (2D_1 + U_{\bar{3}}^*)\sin\theta\} \\
&\quad + \frac{1}{\ln\kappa}\{(2C_2 + C_1 - 2C_1\ln(\rho/\lambda))\cos\theta \\
&\quad + (2D_2 + D_1 - 2D_1\ln(\rho/\lambda))\sin\theta\} + \ldots, \\
u_\theta &\sim \{(-2C_1 - U_{\bar{2}}^*)\sin\theta + (2D_1 + U_{\bar{3}}^*)\cos\theta\} \\
&\quad + \frac{1}{\ln\kappa}\{(-2C_2 + C_1 + 2C_1\ln(\rho/\lambda))\sin\theta \\
&\quad + (2D_2 - D_1 - 2D_1\ln(\rho/\lambda))\cos\theta\} + \ldots, \\
u_z &\sim (-E_1 + U_{\bar{1}}^*) + \frac{1}{\ln\kappa}(-E_2 + E_1\ln(\rho/\lambda)) + \ldots, \\
p &\sim \frac{1}{\ln\kappa}4\rho^{-1}(C_1\cos\theta + D_1\sin\theta) + F_0 + \frac{F_1}{\ln\kappa} + \ldots,
\end{aligned}\right\} \tag{4.11}$$

in the limit of $\rho \to 0$ (i.e. $(\mathbf{u}, p)$ has this form for a point $\mathbf{r}$ which moves in towards the line singularity $\mathbf{r} = \mathbf{R}(s)$ at the point P).

5. Outer expansion

The outer flow field $(\mathbf{u}, p)$ satisfies the creeping motion equations

$$\nabla^2\mathbf{u} - \nabla p = 0, \quad \nabla . \mathbf{u} = 0 \tag{5.1}$$

and would take the value $\mathbf{u} = \mathbf{U}(\mathbf{r})$ if the body S were absent. Thus from the inner boundary condition (4.11) on $(\mathbf{u}, p)$ it would seem reasonable that one would be able to expand $(\mathbf{u}, p)$ in the form

$$\left.\begin{aligned}
\mathbf{u} &= \mathbf{U}(\mathbf{r}) + \frac{\mathbf{u}_1}{\ln\kappa} + \ldots, \\
p &= P(\mathbf{r}) + \frac{p_1}{\ln\kappa} + \ldots.
\end{aligned}\right\} \tag{5.2}$$

At a point P on the line singularity $\mathbf{r} = \mathbf{R}(s)$, it is convenient to take a set of rectangular Cartesian axes with unit base vectors $\mathbf{i}_{\bar{1}}$, $\mathbf{i}_{\bar{2}}$ and $\mathbf{i}_{\bar{3}}$ which lie in the same directions as the $\bar{1}, \bar{2}, \bar{3}$ axes in the inner expansion at P (see § 4). Thus $\mathbf{i}_{\bar{1}}$ lies in the direction of the tangent to $\mathbf{r} = \mathbf{R}(s)$ at P. Since the $\bar{2}$ and $\bar{3}$ axes were arbitrary one may now, for convenience, take $\mathbf{i}_{\bar{2}}$ to lie in the plane containing $\mathbf{i}_{\bar{1}}$ and the velocity $\mathbf{U}(\mathbf{r}) - \mathbf{U}^*(s)$ evaluated at P (see figure 3). Thus the unit vectors $\mathbf{i}_{\bar{1}}$, $\mathbf{i}_{\bar{2}}$ and $\mathbf{i}_{\bar{3}}$ may be written in the form

$$\left.\begin{aligned}
\mathbf{i}_{\bar{1}} &= d\mathbf{R}/ds, \\
\mathbf{i}_{\bar{2}} &= (\mathbf{U} - \mathbf{U}^*).\left[\mathbf{I} - \frac{d\mathbf{R}}{ds}\frac{d\mathbf{R}}{ds}\right] \Big/ \left\{|\mathbf{U} - \mathbf{U}^*|^2 - \left|(\mathbf{U} - \mathbf{U}^*).\left(\frac{d\mathbf{R}}{ds}\right)\right|^2\right\}^{\frac{1}{2}}, \\
\mathbf{i}_{\bar{3}} &= \frac{d\mathbf{R}}{ds} \times (\mathbf{U} - \mathbf{U}^*) \Big/ \left\{|\mathbf{U} - \mathbf{U}^*|^2 - \left|(\mathbf{U} - \mathbf{U}^*).\left(\frac{d\mathbf{R}}{ds}\right)\right|^2\right\}^{\frac{1}{2}},
\end{aligned}\right\} \tag{5.3}$$

where all quantities are evaluated at the point P considered, $\mathbf{I}$ being the idemfactor.

The term in equation (5.2) of order unity (i.e. the flow field $(\mathbf{U}, P)$) must behave near the line $\mathbf{r} = \mathbf{R}(s)$ like the term of order unity in equation (4.11). Hence on $\mathbf{r} = \mathbf{R}(s)$

$$\left.\begin{aligned} U_{\bar 1}(\mathbf{r}) &= -E_1 + U_{\bar 1}^*(s), \\ U_{\bar 2}(\mathbf{r}) &= +2C_1 + U_{\bar 2}^*(s), \\ U_{\bar 3}(\mathbf{r}) &= +2D_1 + U_{\bar 3}^*(s), \\ \text{and} \quad P(\mathbf{r}) &= F_0. \end{aligned}\right\} \tag{5.4}$$

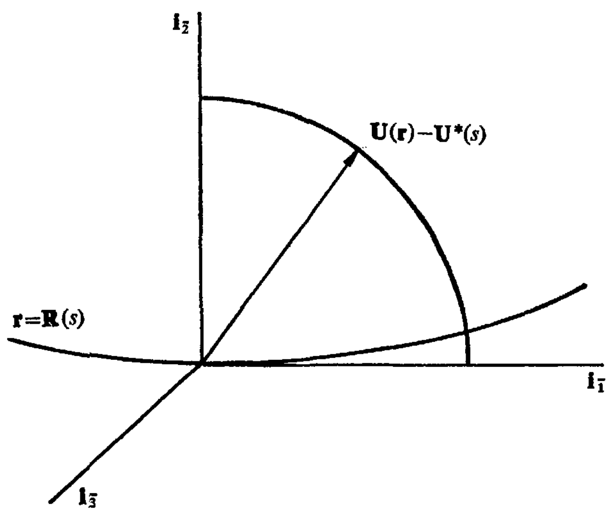

FIGURE 3. System of axes with unit base vectors $\mathbf{i}_{\bar 1}$, $\mathbf{i}_{\bar 2}$, $\mathbf{i}_{\bar 3}$.

Therefore the values of C_1, D_1, E_1 and F_0 are given by

$$\left.\begin{aligned} E_1 &= -(U_{\bar 1} - U_{\bar 1}^*), \\ C_1 &= \tfrac{1}{2}(U_{\bar 2} - U_{\bar 2}^*), \\ D_1 &= \tfrac{1}{2}(U_{\bar 3} - U_{\bar 3}^*), \\ F_0 &= P, \end{aligned}\right\} \tag{5.5}$$

all quantities being evaluated on $\mathbf{r} = \mathbf{R}(s)$. From the definitions of the set of axes, $\bar 1, \bar 2, \bar 3$ given by (5.3), it is seen that

$$\left.\begin{aligned} C_1 &= \tfrac{1}{2}\{|\mathbf{U}(\mathbf{R}) - \mathbf{U}^*(s)|^2 - |\{\mathbf{U}(\mathbf{R}) - \mathbf{U}^*(s)\}.d\mathbf{R}/ds|^2\}^{\frac{1}{2}}, \\ D_1 &= 0, \\ E_1 &= -\{\mathbf{U}(\mathbf{R}) - \mathbf{U}^*(s)\}.d\mathbf{R}/ds, \\ F_0 &= P(\mathbf{R}). \end{aligned}\right\} \tag{5.6}$$

Consider now the matching of terms of order $(1/\ln \kappa)$ in (5.2). It is seen that the flow field $(\mathbf{u}_1, p_1)$ near the line $\mathbf{r} = \mathbf{R}(s)$ must behave like the term of order $(1/\ln \kappa)$ in (4.11). Hence as $\rho \to 0$,

$$\left.\begin{aligned}(u_1)_\rho &\sim (-2C_1\cos\theta)\ln\rho + \{(2C_2 + C_1 + 2C_1\ln\lambda)\cos\theta + (2D_2)\sin\theta\} + \ldots,\\ (u_1)_\theta &\sim (+2C_1\sin\theta)\ln\rho + \{(-2C_2 + C_1 - 2C_1\ln\lambda)\sin\theta + (2D_2)\cos\theta\} + \ldots,\\ (u_1)_z &\sim +E_1\ln\rho + (-E_2 - E_1\ln\lambda),\\ p_1 &\sim 4\rho^{-1}(C_1\cos\theta) + F_1,\end{aligned}\right\} \tag{5.7}$$

where use has been made of the result $D_1 = 0$ (see (5.6)). The singular part of $(\mathbf{u}_1, p_1)$ given by (5.7), (i.e. the terms of order $\ln\rho$ for $\mathbf{u}_1$ and of order ρ^{-1} for p_1) represents a line of force on $\mathbf{r} = \mathbf{R}(s)$ of magnitude $\mathscr{F}^*(s)$ given by

$$\mathscr{F}^*(s) = (8\pi C_1)\,\mathbf{i}_{\bar{2}} - (2\pi E_1)\,\mathbf{i}_{\bar{1}} \tag{5.8}$$

which by equations (5.3) and (5.6) may be written as

$$\mathscr{F}^*(s) = 4\pi\{\mathbf{U}(\mathbf{R}) - \mathbf{U}^*(s)\}.\left\{\mathbf{I} - \frac{1}{2}\frac{d\mathbf{R}}{ds}\frac{d\mathbf{R}}{ds}\right\}. \tag{5.9}$$

Now the flow field $(\mathbf{u}_1, p_1)$ due to this line of force is given by

$$(u_1)_i = \frac{1}{8\pi}\int_0^1 \left\{\frac{\delta_{ij}}{|\mathbf{r} - \hat{\mathbf{R}}|} + \frac{(r_i - \hat{R}_i)(r_j - \hat{R}_j)}{|\mathbf{r} - \hat{\mathbf{R}}|^3}\right\}\mathscr{F}_j^*(\hat{s})\,d\hat{s}, \tag{5.10a}$$

$$p_1 = \frac{1}{4\pi}\int_0^1 \frac{(r_j - \hat{R}_j)}{|\mathbf{r} - \hat{\mathbf{R}}|^3}\mathscr{F}_j^*(\hat{s})\,d\hat{s}, \tag{5.10b}$$

where the integration is taken over the line $0 \leqslant \hat{s} \leqslant 1$ and $\hat{\mathbf{R}}$ represents the value of $\mathbf{r}$ at a point $s = \hat{s}$ on the line of force. The substitution of the value of $\mathscr{F}^*$ from (5.9) into the expressions (5.10) for $(\mathbf{u}_1, p_1)$ yields

$$\left.\begin{aligned}(u_1)_i &= \int_0^1 \frac{1}{2}\left\{\frac{\delta_{ij}}{|\mathbf{r} - \hat{\mathbf{R}}|} + \frac{(r_i - \hat{R}_i)(r_j - \hat{R}_j)}{|\mathbf{r} - \hat{\mathbf{R}}|^3}\right\}\left\{\delta_{jk} - \frac{1}{2}\frac{d\hat{R}_j}{d\hat{s}}\frac{d\hat{R}_k}{d\hat{s}}\right\}\{U_k(\hat{\mathbf{R}}) - U_k^*(\hat{s})\}\,d\hat{s},\\ p_1 &= \int_0^1 \frac{(r_i - \hat{R}_j)}{|\mathbf{r} - \mathbf{R}|^3}\left\{\delta_{jk} - \frac{1}{2}\frac{d\hat{R}_j}{d\hat{s}}\frac{d\hat{R}_k}{d\hat{s}}\right\}\{U_k(\hat{\mathbf{R}}) - U_k^*(\hat{s})\}\,d\hat{s}.\end{aligned}\right\} \tag{5.11}$$

In order to compare this value of $(\mathbf{u}_1, p_1)$ with equation (5.7), one requires to obtain the behaviour of $(\mathbf{u}_1, p_1)$ given by (5.11) near the line of singularity $\mathbf{r} = \mathbf{R}(s)$. Since $\mathbf{u}_1$ and p_1 both become infinite on this line one writes

$$\left.\begin{aligned}(u_1)_i &= J_i + J_i^*,\\ p_1 &= K + K^*,\end{aligned}\right\} \tag{5.12}$$

where J_i and K are the integrals in (5.11) taken over the intervals $(0, s - \epsilon)$ and $(s + \epsilon, 1)$, whilst J_i^* and K^* are the integrals taken over the remaining interval $(s - \epsilon, s + \epsilon)$. The quantity $\epsilon > 0$ is assumed to be arbitrary and very much smaller than unity. Since the integrands in (5.11) only become singular at $\hat{s} = s$ if $\mathbf{r}$ lies on the line singularity, it follows that the integrals J_i and K have integrands with no singularity, although the values of these integrals will tend to infinity as $\epsilon \to 0$. Now if $\epsilon \ll 1$, the integrals J_i^* and K^* may be simplified if one notes that $s \simeq \hat{s}$ in the range of integration. Therefore

$$\left.\begin{aligned} J_i^* &= \frac{1}{2}\left\{\delta_{jk} - \frac{1}{2}\frac{dR_j}{ds}\frac{dR_k}{ds}\right\}\{U_k(\mathbf{R}) - U_k^*(s)\}\int_{s-\epsilon}^{s+\epsilon}\left\{\frac{\delta_{ij}}{|\mathbf{r}-\hat{\mathbf{R}}|} + \frac{(r_i-\hat{R}_i)(r_j-\hat{R}_j)}{|\mathbf{r}-\hat{\mathbf{R}}|^3}\right\}d\hat{s}, \\ K^* &= \left\{\delta_{jk} - \frac{1}{2}\frac{dR_j}{ds}\frac{dR_k}{ds}\right\}\{U_k(\mathbf{R}) - U_k^*(s)\}\int_{s-\epsilon}^{s+\epsilon}\frac{(r_j-\hat{R}_j)}{|\mathbf{r}-\hat{\mathbf{R}}|^3}\,d\hat{s}. \end{aligned}\right\} \tag{5.13}$$

If $\epsilon \ll 1$, the integrals in (5.13) are readily evaluated to give for

$$I_{ij} = \int_{s-\epsilon}^{s+\epsilon}\left\{\frac{\delta_{ij}}{|\mathbf{r}-\hat{\mathbf{R}}|} + \frac{(r_i-\hat{R}_i)(r_j-\hat{R}_j)}{|\mathbf{r}-\hat{\mathbf{R}}|^3}\right\}d\hat{s}, \tag{5.14}$$

the values

$$\left.\begin{aligned} I_{\bar{1}\bar{1}} &= -4\ln\rho + 4\ln\epsilon + 4\ln 2 - 2 + o(1), \\ I_{\bar{2}\bar{2}} &= -2\ln\rho + 2\ln\epsilon + 2\ln 2 + 2\cos^2\theta + o(1), \\ I_{\bar{3}\bar{3}} &= -2\ln\rho + 2\ln\epsilon + 2\ln 2 + 2\sin^2\theta + o(1), \\ I_{\bar{2}\bar{3}} &= I_{\bar{3}\bar{2}} = 2\sin\theta\cos\theta + o(1), \\ I_{ij} &= o(1) \quad \text{for all other } i, j \quad \text{as} \quad \rho \to 0. \end{aligned}\right\} \tag{5.15}$$

Also for

$$I_j = \int_{s-\epsilon}^{s+\epsilon}\frac{(r_j-\hat{R}_j)}{|\mathbf{r}-\hat{\mathbf{R}}|^3}\,ds \tag{5.16}$$

one has in the limit of $\rho \to 0$ (and $\epsilon \ll 1$)

$$\left.\begin{aligned} I_{\bar{1}} &= o(1), \\ I_{\bar{2}} &= +2\rho^{-1}\cos\theta + o(1), \\ I_{\bar{3}} &= +2\rho^{-1}\sin\theta + o(1), \end{aligned}\right\} \tag{5.17}$$

the values of I_{ij} and I_j being taken relative to the $\bar{1}, \bar{2}, \bar{3}$ set of axes. By substituting these values of the integrals I_{ij} and I_j into the expressions (5.13) and also noting that relative to the $\bar{1}, \bar{2}, \bar{3}$ set of axes

$$dR_j/ds = \delta_{j1} \quad \text{and} \quad (U_{\bar{3}} - U_{\bar{3}}^*) = 0,$$

it is seen that the value of $(\mathbf{u}_1, p_1)$ for $\rho \to 0$ may by (5.12) be written as

$$\left.\begin{aligned} (u_1)_{\bar{1}} &\sim (U_{\bar{1}} - U_{\bar{1}}^*)(-\ln\rho + \ln\epsilon + \ln 2 - \tfrac{1}{2}) + J_{\bar{1}}, \\ (u_1)_{\bar{2}} &\sim (U_{\bar{2}} - U_{\bar{2}}^*)(-\ln\rho + \ln\epsilon + \ln 2 + \cos^2\theta) + J_{\bar{2}}, \\ (u_1)_{\bar{3}} &\sim (U_{\bar{2}} - U_{\bar{2}}^*)\sin\theta\cos\theta + J_{\bar{3}}, \\ p_1 &\sim (U_{\bar{2}} - U_{\bar{2}}^*)\,2\rho^{-1}\cos\theta + K. \end{aligned}\right\} \tag{5.18}$$

Thus in terms of the polar co-ordinates (ρ, θ, z), the velocity field $(\mathbf{u}_1, p_1)$ behaves like

$$\left.\begin{aligned} (u_1) &\sim -\{(U_{\bar{2}} - U_{\bar{2}}^*)\cos\theta\}\ln\rho + (U_{\bar{2}} - U_{\bar{2}}^*)\{(\ln\epsilon + \ln 2 + 1)\cos\theta\} \\ &\qquad + J_{\bar{2}}\cos\theta + J_{\bar{3}}\sin\theta, \\ (u_1)_\theta &\sim +\{(U_{\bar{2}} - U_{\bar{2}}^*)\sin\theta\}\ln\rho + (U_{\bar{2}} - U_{\bar{2}}^*)\{(-\ln\epsilon - \ln 2)\sin\theta\} \\ &\qquad - J_{\bar{2}}\sin\theta + J_{\bar{3}}\cos\theta, \\ (u_1)_z &\sim -(U_{\bar{1}} - U_{\bar{1}}^*)\ln\rho + (U_{\bar{1}} - U_{\bar{1}}^*)(\ln\epsilon + \ln 2 - \tfrac{1}{2}) + J_{\bar{1}}, \\ p_1 &\sim (U_{\bar{2}} - U_{\bar{2}}^*)\,2\rho^{-1}\cos\theta + K. \end{aligned}\right\} \tag{5.19}$$

Comparing these equations with the asymptotic form of $(\mathbf{u}_1, p_1)$ near $\mathbf{r} = \mathbf{R}(s)$ given by (5.7), it is observed that the terms in $\mathbf{u}_1$ of order $\ln \rho$ and the term in p_1 of order ρ^{-1} are equal (as they must be since solutions have already been matched to this order). Equating terms in $\mathbf{u}_1$ of order ρ^0 and making use of the values of C_1, D_1 and E_1 given by equation (5.5) one obtains the values of the constants C_2, D_2, and E_2 as

$$\left.\begin{aligned} C_2 &= \tfrac{1}{2}(U_{\bar{2}} - U^*_{\bar{2}})\,(\tfrac{1}{2} + \ln 2 - \ln\lambda + \ln\epsilon) + \tfrac{1}{2}J_{\bar{2}},\\ D_2 &= \tfrac{1}{2}J_{\bar{3}},\\ E_2 &= (U_{\bar{1}} - U^*_{\bar{1}})\,(\tfrac{1}{2} - \ln 2 + \ln\lambda - \ln\epsilon) - J_{\bar{1}}. \end{aligned}\right\} \tag{5.20}$$

Thus combining equations (4.9), (5.5) and (5.21), the values of $C(\kappa)$, $D(\kappa)$ and $E(\kappa)$ are obtained as

$$\left.\begin{aligned} C(\kappa) &= (\ln\kappa)^{-1}\tfrac{1}{2}(U_{\bar{2}} - U^*_{\bar{2}})\\ &\qquad + (\ln\kappa)^{-2}\{\tfrac{1}{2}(U_{\bar{2}} - U^*_{\bar{2}})\,(\tfrac{1}{2} + \ln 2 - \ln\lambda + \ln\epsilon) + \tfrac{1}{2}J_{\bar{2}}\} + \ldots,\\ D(\kappa) &= (\ln\kappa)^{-2}\tfrac{1}{2}J_{\bar{3}},\\ E(\kappa) &= -(\ln\kappa)^{-1}(U_{\bar{1}} - U^*_{\bar{1}})\\ &\qquad + (\ln\kappa)^{-2}\{(U_{\bar{1}} - U^*_{\bar{1}})\,(\tfrac{1}{2} - \ln 2 + \ln\lambda - \ln\epsilon) - J_{\bar{1}}\} + \ldots. \end{aligned}\right\} \tag{5.21}$$

6. Force on body

From the value of the inner velocity field given by equations (4.8), the force per unit length $\mathscr{F}$ exerted by the fluid on the body may be evaluated. Relative to the $\bar{1}$, $\bar{2}$, $\bar{3}$ set of axes this quantity $\mathscr{F}$ may therefore be shown to have the components $(2\pi E(\kappa), -8\pi C(\kappa), -8\pi D(\kappa))$ so that one may write

$$\mathscr{F}/2\pi = E(\kappa)\,\mathbf{i}_{\bar{1}} - 4C(\kappa)\,\mathbf{i}_{\bar{2}} - 4D(\kappa)\,\mathbf{i}_{\bar{3}}. \tag{6.1}$$

Substituting the values of $C(\kappa)$, $D(\kappa)$ and $E(\kappa)$ from equations (5.21) and by making use of the definitions of the unit vectors $\mathbf{i}_{\bar{1}}$, $\mathbf{i}_{\bar{2}}$, $\mathbf{i}_{\bar{3}}$ (see equation (5.3)), one may put (6.1) in the form

$$\begin{aligned} \mathscr{F}/2\pi = &\left[\frac{(\mathbf{U}-\mathbf{U}^*)}{\ln\kappa} + \frac{\mathbf{J} + (\mathbf{U}-\mathbf{U}^*)\ln(2\epsilon/\lambda)}{(\ln\kappa)^2}\right]\cdot\left[\frac{d\mathbf{R}}{ds}\frac{d\mathbf{R}}{ds} - 2\mathbf{I}\right]\\ &+ \frac{\tfrac{1}{2}(\mathbf{U}-\mathbf{U}^*)}{(\ln\kappa)^2}\cdot\left[3\frac{d\mathbf{R}}{ds}\frac{d\mathbf{R}}{ds} - 2\mathbf{I}\right] + O\left\{\frac{1}{(\ln\kappa)^3}\right\}, \end{aligned} \tag{6.2}$$

where $\mathbf{J}$ is, by definition, a vector given by

$$\begin{aligned} J_i = \frac{1}{2}\left[\int_0^{s-\epsilon} + \int_{s+\epsilon}^1\right] &\left\{\frac{\delta_{ij}}{|\mathbf{R}-\hat{\mathbf{R}}|} + \frac{(R_i - \hat{R}_i)(R_j - \hat{R}_j)}{|\mathbf{R}-\hat{\mathbf{R}}|^3}\right\}\\ &\times\left\{\delta_{jk} - \frac{1}{2}\frac{d\hat{R}_j}{d\hat{s}}\frac{d\hat{R}_k}{d\hat{s}}\right\}\{U_k(\hat{\mathbf{R}}) - U^*_k(\hat{s})\}\,d\hat{s}, \end{aligned} \tag{6.3}$$

where $\mathbf{R}$ is the value of $\mathbf{r}$ at the point on the centre-line under consideration and $\hat{\mathbf{R}}$ is the value of $\mathbf{r}$ at the point on the centre-line with $s = \hat{s}$. Since the above equations (6.2) and (6.3) for the hydrodynamic force per unit length acting on the body are of vectorial form, one may consider them relative to any rectangular Cartesian co-ordinate system which one may choose. The value of ϵ appearing in equations (6.2) and (6.3) is arbitrary, satisfying only the inequality $0 < \epsilon \ll 1$. However it may be shown that at $\epsilon \to 0$, the value of $\mathbf{J}$ given by (6.3) is of the form

$$\mathbf{J} \sim -\{\mathbf{U}(\mathbf{R}) - \mathbf{U}^*(s)\} \ln \epsilon + O(1), \tag{6.4}$$

so that when this value of $\mathbf{J}$ is substituted into (6.2), the resulting expression is finite and independent of ϵ in the limit of $\epsilon \to 0$. Thus the final expression for the force per unit length acting on S is independent of ϵ. The total force $\mathbf{F}$ and torque $\mathbf{G}$ acting on the body S relative to any origin is given by

$$\left.\begin{aligned} \mathbf{F} &= \int_0^1 \mathscr{F}(s)\, ds, \\ \mathbf{G} &= \int_0^1 \mathbf{R}(s) \times \mathscr{F}(s)\, ds, \end{aligned}\right\} \tag{6.5}$$

where $\mathbf{R}(s)$ is the vector from the origin to the position s on the body centre-line.

From the general theory it is seen that (6.2) is valid at all points along the body only if the cross-sectional radius of the body is a continuous function of length along the centre-line, i.e. one requires $\lambda(s)$ to have the properties

$$\left.\begin{aligned} &\text{(i) } \lambda(s) \text{ is continuous for } 0 \leqslant s \leqslant 1, \\ &\text{(ii) } \lambda(0) = \lambda(1) = 0. \end{aligned}\right\} \tag{6.6}$$

However if $\lambda(s)$ satisfies instead only the properties

$$\left.\begin{aligned} &\text{(iii) } \lambda(s) \text{ is piecewise continuous in } 0 \leqslant s \leqslant 1, \\ &\text{(iv) } \lambda(s) \text{ has a finite number of discontinuities in } 0 \leqslant s \leqslant 1, \end{aligned}\right\} \tag{6.7}$$

then one would expect the equation (6.2) to be valid except for intervals in s around each point of discontinuity, these intervals each having a width of order κ. However, the quantity $\mathbf{J}$ appearing in (6.2) if evaluated by the use of the integral (6.3) would then possess an error of order κ since the integrand in (6.3) is valid only outside of intervals of width κ (in $\hat{s}$) around each point of discontinuity of $\lambda(\hat{s})$. Thus to the order considered the equation (6.2) with $\mathbf{J}$ given by (6.3) gives the force per unit length acting on the body except within the intervals of width κ about each discontinuity in $\lambda(s)$. Therefore the total force and torque acting on the body are given by equations (6.5) to the order in κ considered.

The theory given in §§ 3, 4 and 5 may be easily modified to include the case of (i) the interaction of two long slender bodies placed in an undisturbed flow $\mathbf{U}(\mathbf{r})$ and (ii) the interaction of a single long slender body placed in an undisturbed flow $\mathbf{U}(\mathbf{r})$, there being some solid walls W present. The results for these two cases are given below. However, their derivation will not be given since they may readily be deduced by repeating, for these cases, the general theory.

(i) *Interaction of two bodies*

Consider two long slender bodies S and S' of lengths l and l' respectively placed in a fluid undergoing an undisturbed flow $\mathbf{U}(\mathbf{r})$. Use l as unit of length in dimensionless system and define distances along the body centre-line by the quantities s and s' so that

$$0 \leqslant s \leqslant 1, \quad 0 \leqslant s' \leqslant (l'/l).$$

Then the force $\mathscr{F}$ per unit length acting on the body S at a point $\mathbf{r} = \mathbf{R}(s)$ is given by the equation (6.2) with the vector $\mathbf{J}$ given by

$$J_i = \frac{1}{2}\left[\int_0^{s-\epsilon} + \int_{s+\epsilon}^1\right]\left\{\frac{\delta_{ij}}{|\mathbf{R}-\hat{\mathbf{R}}|} + \frac{(R_i - \hat{R}_i)(R_j - \hat{R}_j)}{|\mathbf{R}-\hat{\mathbf{R}}|^3}\right\} \times \left\{\delta_{jk} - \frac{1}{2}\frac{d\hat{R}_j}{d\hat{s}}\frac{d\hat{R}_k}{d\hat{s}}\right\}\{U_k(\hat{\mathbf{R}}) - U_k^*(\hat{s})\}\,d\hat{s}$$
$$+\frac{1}{2}\int_0^{l'/l}\left\{\frac{\delta_{ij}}{|\mathbf{R}-\mathbf{R}'|} + \frac{(R_i - R'_i)(R_j - R'_j)}{|R-R'|^3}\right\}\left\{\delta_{jk} - \frac{1}{2}\frac{dR'_j}{ds'}\frac{dR'_k}{ds'}\right\}\{U_k(\mathbf{R}') - U_k^*(s')\}\,ds', \tag{6.8}$$

where $\mathbf{R}'$ is the value of $\mathbf{r}$ at the point $s = s'$ on the centre-line of the body S', $\mathbf{U}^*(s')$ being the velocity of the material of the body S' at such a point. One may write down an expression for the force per unit length acting on S' by interchanging the roles played by S and S'. In a similar manner one may write down the force on any long slender body S in the presence of any (finite) number of other long slender bodies.

(ii) *Interaction of body with wall*

Consider a single long slender body S placed in a fluid undergoing an undisturbed flow $\mathbf{U}(\mathbf{r})$ in the presence of a system of walls W. Then the flow $\mathbf{U}(\mathbf{r})$ itself must be zero on all such walls. Then the force $\mathscr{F}$ per unit length acting on S is given by (6.2) with the vector $\mathbf{J}$ now given by the expression

$$J_i = \frac{1}{2}\left[\int_0^{s-\epsilon} + \int_{s+\epsilon}^1\right] f_{ij}(\mathbf{R}, \hat{\mathbf{R}})\left\{\delta_{jk} - \frac{1}{2}\frac{d\hat{R}_j}{d\hat{s}}\frac{d\hat{R}_k}{d\hat{s}}\right\}\{U_k(\hat{\mathbf{R}}) - U_k^*(\hat{s})\}\,d\hat{s}, \tag{6.9}$$

where $f_{ij}(\mathbf{R}, \hat{\mathbf{R}})$ is the Greens function for creeping motion flow in the presence of the walls W, i.e. $f_{ij}(\mathbf{r}, \hat{\mathbf{r}})$ is defined by the equations

$$\left.\begin{aligned} f_{ij,kk} - g_{j,i} + \delta_{ij}\delta(\mathbf{r}-\hat{\mathbf{r}}) &= 0,\\ f_{ij,i} &= 0,\end{aligned}\right\} \tag{6.10}$$

with the boundary condition

$$f_{ij} = 0 \quad \text{on} \quad W, \tag{6.11}$$

g_j being the pressure field corresponding to f_{ij} and $\delta(\mathbf{r}-\hat{\mathbf{r}})$ the Dirac delta function.

Similarly by replacing the terms

$$\left\{\frac{\delta_{ij}}{|\mathbf{R}-\hat{\mathbf{R}}|} + \frac{(R_i - \hat{R}_i)(R_j - \hat{R}_j)}{|\mathbf{R}-\hat{\mathbf{R}}|^3}\right\} \quad \text{and} \quad \left\{\frac{\delta_{ij}}{|\mathbf{R}-\mathbf{R}'|} + \frac{(R_i - R'_i)(R_j - R'_j)}{|\mathbf{R}-\mathbf{R}'|^3}\right\}$$

in (6.8) by the functions $f_{ij}(\mathbf{R}, \hat{\mathbf{R}})$ and $f_{ij}(\mathbf{R}, \mathbf{R}')$ respectively, one obtains the value of the force per unit length acting upon a body S in the presence of both another long slender body S' and some walls W.

7. Examples of long slender bodies in translation

The results given in the previous section for the force acting on a long slender body will now be used to determine the resistance to translation of a long slender body for which the body centre-line is straight. Thus consider such a body and take a set of rectangular Cartesian axes (r_1, r_2, r_3) with origin O at the point on the centre-line of the body midway between the ends. Take the r_1 axis in the direction of the centre-line (see figure 4) and let the length of the body be $2a$.

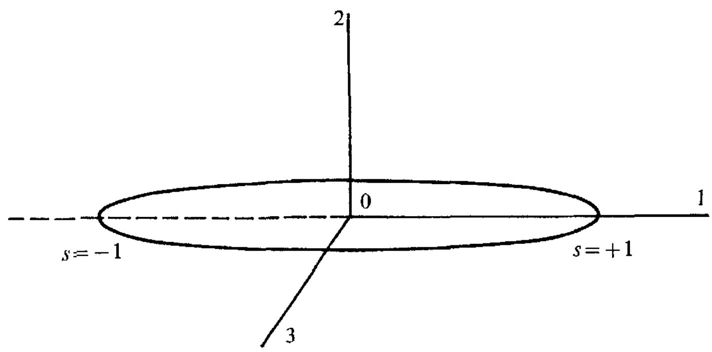

FIGURE 4. System of axes for long slender body with straight centre-line.

Let b now be the cross-sectional radius of the body at the origin so that

$$\lambda(0) = 1. \tag{7.1}$$

Then using quantities made dimensionless with respect to a, so that $\kappa = b/a$, it is seen that the integrals in (6.3) and (6.5) are to be taken over the range

$$-1 \leqslant s \leqslant +1.$$

Consider the body at rest in a fluid undergoing a uniform translation in the 1-direction, so that one may take

$$\mathbf{U} = (1, 0, 0), \quad \mathbf{U}^* = 0. \tag{7.2}$$

The centre-line $\mathbf{r} = \mathbf{R}(s)$ of the body is given by

$$R_i(s) = s\delta_{i1} \quad \text{for} \quad -1 \leqslant s \leqslant +1. \tag{7.3}$$

Hence

$$dR_i/ds = \delta_{i1}. \tag{7.4}$$

Substitution of these expressions into (6.3) yields

$$J_i = \delta_{i1}\{-\ln \epsilon + \tfrac{1}{2}\ln(1-s^2)\}. \tag{7.5}$$

One may now evaluate the force per unit length $\mathscr{F}$ acting on the body from (6.2). Thus one may obtain

$$\mathscr{F} = (\mathscr{F}_1, 0, 0), \tag{7.6}$$

where

$$\mathscr{F}_1/2\pi = -\frac{1}{\ln \kappa} + \frac{1}{(\ln \kappa)^2}\left\{\tfrac{1}{2} - \ln 2 - \tfrac{1}{2}\ln\left(\frac{1-s^2}{\lambda^2}\right)\right\} + O\left\{\frac{1}{(\ln \kappa)^3}\right\}. \tag{7.7}$$

By substituting this expression into (6.5) one may obtain the total force $\mathbf{F}$ on the body in *dimensional* form as

$$\mathbf{F} = (F_1, 0, 0), \tag{7.8}$$

where

$$F_1 = 4\pi\mu a U\left[\frac{1}{\ln a/b}+\frac{1}{(\ln a/b)^2}\left\{\tfrac{1}{2}-\ln 2-\frac{1}{4}\int_{-1}^{+1}\ln\left(\frac{1-s^2}{\lambda^2}\right)ds\right\}+O\left\{\frac{1}{(\ln a/b)^3}\right\}\right]. \tag{7.9}$$

This equation, which is identical to that obtained by Tillett (1970) and Batchelor (1970), may be written in the alternative form

$$F_1 = \frac{4\pi\mu a U}{\ln(2a/b)+C_1}+O\left\{\frac{\mu a U}{(\ln a/b)^3}\right\}, \tag{7.10}$$

where C_1 depends on the body shape and is given by

$$C_1 = -\frac{1}{2}+\frac{1}{4}\int_{-1}^{+1}\ln\left(\frac{1-s^2}{\lambda^2}\right)ds. \tag{7.11}$$

As examples, consider the following cases:

(i) A *circular cylinder* for which

$$\lambda(s) = 1 \quad \text{for} \quad -1 \leqslant s \leqslant +1. \tag{7.12a}$$

This possesses a value of C_1 given by

$$C_1 = -\tfrac{3}{2}+\ln 2 = -0{\cdot}80685. \tag{7.12b}$$

(ii) A *spheroid* for which

$$\lambda(s) = (1-s^2)^{\frac{1}{2}} \quad \text{for} \quad -1 \leqslant s \leqslant +1. \tag{7.13a}$$

This possesses a value of C_1 given by

$$C_1 = -\tfrac{1}{2}. \tag{7.13b}$$

(iii) A *double cone* for which

$$\lambda(s) = 1+s \quad \text{for} \quad -1 \leqslant s \leqslant 0,$$
$$= 1-s \quad \text{for} \quad 0 \leqslant s \leqslant +1. \tag{7.14a}$$

This possesses a value of C_1 given by

$$C_1 = -\tfrac{1}{2}+\ln 2 = +0{\cdot}19315. \tag{7.14b}$$

It should be noted that the value of C_1 for case (ii) is greater than that for case (i) so that the resistance force **F** for a given value of a/b is greater for the circular cylinder (case (i)) than for the spheroid (case (ii)), while similarly the resistance force for the spheroid is greater than that for the double cone (case (iii)). This is what one would expect. The result for the spheroid given above agrees with Oberbeck's solution (see § 2), whilst the result for a circular cylinder agrees with that obtained by Broersma (see equation (2.9)). However the present method yields a value of C_1 analytically as being equal to $-\frac{3}{2}+\ln 2$ or $-0{\cdot}80685$ whereas Broersma was able to obtain $C_1 = -0{\cdot}81$ only after much numerical computation. [Burgers gave $C_1 = -0{\cdot}72$ for this case.] The present method gives a solution which is a true asymptotic expansion in the parameter b/a and so gives an estimate of the error in the expression for the force **F** (see equation (7.10)).

Consider now the long slender body at rest in a fluid undergoing a uniform translation in the direction of the r_2 axis (see figure 4), so that now one has

$$\mathbf{U} = (0,1,0), \quad \mathbf{U}^* = 0. \tag{7.15}$$

Since (7.3) and (7.4) are still valid, the equation (6.3) for $\mathbf{J}$ yields for the present case

$$J_i = \delta_{i2}\{-\ln\epsilon + \tfrac{1}{2}\ln(1-s^2)\}. \tag{7.16}$$

The equation (6.2) for the force $\mathscr{F}$ per unit length acting on the body gives

$$\mathscr{F} = (0, \mathscr{F}_2, 0), \tag{7.17}$$

where
$$\mathscr{F}_2/2\pi = -\frac{2}{\ln\kappa} + \frac{1}{(\ln\kappa)^2}\left\{-1-2\ln 2 - \ln\left(\frac{1-s^2}{\lambda^2}\right)\right\} + O\left\{\frac{1}{(\ln\kappa)^3}\right\}. \tag{7.18}$$

The substitution of this expression into (6.5) yields the total force $\mathbf{F}$ and torque $\mathbf{G}$ on the body. These may be written in *dimensional* form as

$$\mathbf{F} = (0, F_2, 0), \tag{7.19}$$

$$\mathbf{G} = (0, 0, G_3), \tag{7.20}$$

where
$$F_2 = \frac{8\pi\mu a U}{\ln(2a/b) + C_2} + O\left\{\frac{\mu a U}{(\ln(a/b))^3}\right\}, \tag{7.21}$$

$$G_3 = \frac{2\pi D\mu a^2 U}{\{\ln(2a/b)\}^2} + O\left\{\frac{\mu a^2 U}{(\ln(a/b))^3}\right\}, \tag{7.22}$$

C_2 and D being constants dependent upon body shape and given by

$$C_2 = +\frac{1}{2} + \frac{1}{4}\int_{-1}^{+1} \ln\left(\frac{1-s^2}{\lambda^2}\right) ds, \tag{7.23}$$

$$D = -\int_{-1}^{+1} s\ln\left(\frac{1-s^2}{\lambda^2}\right) ds. \tag{7.24}$$

It should be noted that the value of C_2 given here is just $C_1 + 1$, the quantity C_1 being involved in the formula for the force F for uniform translation of fluid parallel to body axis (see (7.10) and (7.11)). The values of D and G_3 are zero for bodies like a circular cylinder and spheroid which possess fore-aft symmetry. As an example of a non-symmetric body consider the cone given by

$$\lambda(s) = 1 - s \quad \text{for} \quad -1 \leqslant s \leqslant 1. \tag{7.25}$$

Substitution of this value of $\lambda(s)$ into (7.23) and (7.24) gives

$$C_2 = +\tfrac{1}{2}, \quad D = -2, \tag{7.26}$$

so that the force and torque acting on the cone about the origin O is

$$F_2 = \frac{8\pi\mu a U}{\ln(2a/b) + 0{\cdot}5} + O\left\{\frac{\mu a U}{(\ln a/b)^3}\right\},$$

$$G_3 = -\frac{4\pi\mu a^2 U}{\{\ln(2a/b)\}^2} + O\left\{\frac{\mu a^2 U}{(\ln a/b)^3}\right\}, \tag{7.27}$$

so that the resultant force passes through a point P on the body centre-line at a distance $a/(2\ln(2a/b))$ from the mid-point 0, the point P being nearer the base than the apex of the cone.

As an example of the resistance to motion of a body whose centre-line is not straight, consider a long slender body with its centre-line bent in an arc of a circle. Take rectangular Cartesian axes r_1, r_2, r_3 with the r_1 and r_2 axes lying in the plane of the centre-line, the origin of co-ordinates being the centre of curvature of the body. Let the radius of curvature of the body centre-line be a and let b be the characteristic cross-sectional radius of the body. Then relative to polar axes (r, θ), the body centre-line is

$$r = 1, \quad \theta_0 \leqslant \theta \leqslant \theta_1,$$

the value of θ being θ_0 and θ_1 at the ends of the body (see figure 5). Suppose such a body is at rest in a fluid undergoing a uniform translation in the direction of the r_1 axis. Then

$$U_i = \delta_{i1}, \quad U_i^* = 0, \tag{7.28}$$

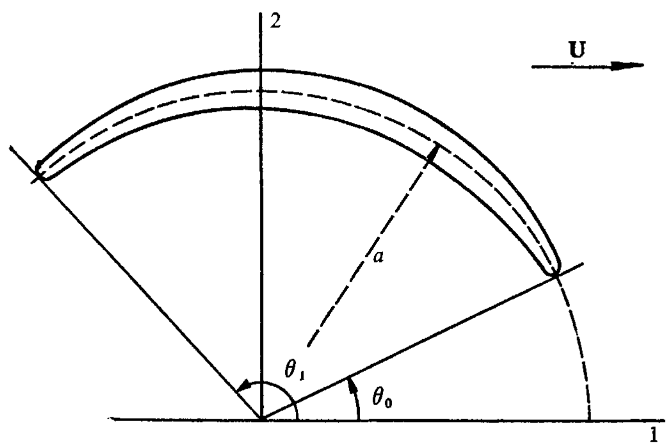

FIGURE 5. Long slender body bent into an arc of a circle.

and taking $s = \theta$, one has

$$\begin{aligned} R_i &= \delta_{i1}\cos\theta + \delta_{i2}\sin\theta, \\ \partial R_i/\partial s &= -\delta_{i1}\sin\theta + \delta_{i2}\cos\theta. \end{aligned} \tag{7.29}$$

Substituting these values into (6.3) for J, one obtains

$$\mathbf{J} = (J_1, J_2, 0), \tag{7.30}$$

where

$$\begin{aligned} J_1 &= \tfrac{1}{4}\{2\ln[\tan\tfrac{1}{4}(\theta-\theta_0)\tan\tfrac{1}{4}(\theta_1-\theta)] - \sin\tfrac{1}{2}(\theta+\theta_0)\sin\theta_0 \\ &\quad - \sin\tfrac{1}{2}(\theta+\theta_1)\sin\theta_1 - 4\ln\epsilon + 8\ln 2 + 2\sin^2\theta\}, \\ J_2 &= \tfrac{1}{4}\{\sin\theta_0\cos\tfrac{1}{2}(\theta+\theta_0) + \sin\theta_1\cos\tfrac{1}{2}(\theta+\theta_1) - \sin 2\theta\}. \end{aligned} \tag{7.31}$$

With this value of $\mathbf{J}$, the formula (6.2) for the force per unit length acting on the body $\mathscr{F}$ has a component in the r_1 direction given by

$$\mathscr{F}_1/2\pi = \frac{A}{\ln\kappa} + \frac{B}{(\ln\kappa)^2}, \tag{7.32}$$

where

$$\begin{aligned} A &= \sin^2\theta - 2, \\ B &= \tfrac{1}{4}(\sin^2\theta - 2)\{2\ln[\tan\tfrac{1}{4}(\theta-\theta_0)\tan\tfrac{1}{4}(\theta_1-\theta)] - \sin\theta_0\sin\tfrac{1}{2}(\theta+\theta_0) \\ &\quad - \sin\theta_1\sin\tfrac{1}{2}(\theta+\theta_1) + 12\ln 2 + 2 + 2\sin^2\theta - 4\ln\lambda\} \\ &\quad - \tfrac{1}{4}\sin\theta\cos\theta\{\sin\theta_0\cos\tfrac{1}{2}(\theta+\theta_0) + \sin\theta_1\cos\tfrac{1}{2}(\theta+\theta_1) - \sin 2\theta\} + \sin^2\theta. \end{aligned} \tag{7.33}$$

Substitution of this result into (6.5) yields the r_1 component of the total force **F** acting on the body. Thus in *dimensional* form the resistance force F_1 is given by

$$F_1 = 2\pi\mu a U\left\{\frac{A^*}{\ln a/b}+\frac{B^*}{(\ln a/b)^2}+O\left(\frac{1}{(\ln a/b)^3}\right)\right\}, \tag{7.34}$$

where

$$\begin{aligned} A^* &= \tfrac{1}{2}\{3(\theta_1-\theta_0)+\cos(\theta_0+\theta_1)\sin(\theta_1-\theta_0)\},\\ B^* &= -\tfrac{1}{2}\cos(\theta_0+\theta_1)\sin(\theta_1-\theta_0)\ln(\tan\tfrac{1}{4}(\theta_1-\theta_0))-6\int_0^{\tan\frac{1}{4}(\theta_1-\theta_0)}\frac{\ln t\,dt}{1+t^2}\\ &\quad+\tfrac{1}{2}\cos(\theta_0+\theta_1)\sin\tfrac{1}{2}(\theta_1-\theta_0)-\tfrac{1}{12}\sin\tfrac{3}{2}(\theta_1-\theta_0)-\tfrac{3}{4}\sin\tfrac{1}{2}(\theta_1-\theta_0)\\ &\quad-\tfrac{1}{2}(1+9\ln 2)(\theta_1-\theta_0)-\tfrac{1}{2}(1+3\ln 2)\cos(\theta_0+\theta_1)\sin(\theta_1-\theta_0)\\ &\quad+\frac{1}{2}\int_{\theta_0}^{\theta_1}(3+\cos 2\theta)(\ln\lambda)\,d\theta. \end{aligned} \tag{7.35}$$

However, it should be noted that in addition to the resistance component of force given above there is, in general, a component of lift force in the r_2-direction. For the case of a completely circular body (like a ring) for which $\theta_1-\theta_0 = 2\pi$, the value of F_1 is given by

$$F_1 = \frac{3\pi\mu l U}{\ln(l/b)+K}, \tag{7.36}$$

where $l = 2\pi a$ is the length of the body and K is a constant given by

$$K = -\ln 2\pi+\tfrac{1}{3}+3\ln 2-\frac{1}{6\pi}\int_{\theta_0}^{\theta_0+2\pi}(3+\cos 2\theta)\ln\lambda\,d\theta. \tag{7.37}$$

For a long thin spheroid bent in the form of a circle, one has

$$\lambda = \left\{1-\left(\frac{-\pi-\theta_0+\theta}{\pi}\right)^2\right\}^{\frac{1}{2}} \quad \text{for} \quad \theta_0 \leqslant \theta \leqslant \theta_0+2\pi, \tag{7.38}$$

$\theta = \theta_0$ representing the ends of the body. This gives a value of the constant K as

$$K = -\ln(\tfrac{1}{2}\pi)+\tfrac{4}{3}+\tfrac{1}{12}\pi\,\mathrm{Si}\,(4\pi)\cos 2\theta_0 \tag{7.39}$$

where $\mathrm{Si}\,(z)$ is the sine integral defined by

$$\mathrm{Si}\,(z) = \int_0^z \frac{\sin t}{t}\,dt. \tag{7.40}$$

Thus

$$K = 0{\cdot}8817+0{\cdot}0396\cos 2\theta_0. \tag{7.41}$$

Similarly for a long slender circular cylinder bent in a circle (i.e. a uniform circular ring), one has

$$\lambda = 1 \quad \text{for} \quad \theta_0 \leqslant \theta \leqslant \theta_0+2\pi,$$

which gives

$$K = -\ln(\tfrac{1}{4}\pi)+\tfrac{1}{3} = +0{\cdot}5749. \tag{7.42}$$

It should be noted that this result is independent of θ_0 as it must be by symmetry. However, for the spheroid bent into the form of a circle (see (7.36) and (7.41)), the resistance force F_1 is dependent upon θ_0, i.e. the force F_1 depends upon the body orientation relative to fluid motion. This particular case of a spheroid bent in the form of a circle was investigated by Tchen (1954) using Burgers' method and derived for the force F_1 a formula identical to (7.36) but with K having for all θ_0, a value of $-2{\cdot}09$. This value, as one would expect, differs from the correct value given by (7.41).

8. Uses and limitations of general theory

The general theory given in §§ 3 to 6 gives the force per unit length $\mathscr{F}$ acting on a long slender body when it is placed in some given undisturbed Stokes flow. The results, given by equations (6.2) and (6.3), may be used to solve a variety of interesting problems. However these problems will not be solved here but further discussion of them will be given in future papers. Thus one may consider the following problems:

(i) The force and torque acting upon long slender rigid bodies placed in a given Stokes flow may be found by direct application of equations (6.2), (6.3) and (6.5). Examples for bodies with straight centre-lines and for bodies with curved centre-lines were given in § 7. One may also make use of the linearity of the creeping motion equations to show, for example, that the total force $\mathbf{F}$ acting upon a circular cylinder of finite length at rest in a uniform fluid flow of velocity $\mathbf{U}$ is given by

$$F_i = \mu a A_{ij} U_j, \tag{8.1}$$

where, relative to the r_1, r_2, r_3 axes given in figure 4, A_{ij} is a tensor given by

$$\left.\begin{aligned} A_{11} &= \frac{4\pi}{\ln(2a/b)-\frac{3}{2}+\ln 2}+O\left\{\frac{1}{(\ln a/b)^3}\right\},\\ A_{22} = A_{33} &= \frac{8\pi}{\ln(2a/b)-\frac{1}{2}+\ln 2}+O\left\{\frac{1}{(\ln a/b)^3}\right\},\\ A_{ij} &= 0 \quad \text{otherwise.} \end{aligned}\right\} \tag{8.2}$$

(ii) One may use the general theory to find the disturbance produced by a long slender body in shear flow. This is necessary if one wishes to derive the equivalent viscosity and rheological behaviour of a suspension of rigid rod-like particles for the case in which the effects of particle collisions are neglected.

(iii) The equations (6.2) and (6.3) may be used to derive the force per unit length on a long slender body for the case in which the body is flexible. One may therefore determine how such a body with known elastic properties would bend if it were placed in a given Stokes flow. For example one could derive the conditions for buckling of a slightly flexible body placed in shear flow. Such a buckling was observed experimentally by Forgacs & Mason (1959*a*, *b*).

(iv) From equations (6.2) and (6.8) the motion of two rod-like particles under their mutual interaction may be determined when (*a*) they are sedimenting in a fluid at rest or (*b*) they interact with one another in a fluid undergoing shear flow. This latter case is useful for the consideration of particle interactions in a suspension of rod-like particles undergoing shear flow. Experimental observations of the behaviour of particles in such a suspension have been made by Anczurowski & Mason (1967*a*, *b*) and Anczurowski, Cox & Mason (1967).

(v) The equations (6.2) and (6.9) may be used to determine the behaviour of a long slender body sedimenting near a plane vertical solid wall, the Greens function $f_{ij}(\mathbf{R},\mathbf{R})$ being readily obtainable for this case.

(vi) At zero Reynolds number, the orientation of a rigid rod-like particle in shear flow undergoes a periodic motion, the motion itself being dependent upon a quantity C called the orbit constant which is determined by the initial orientation of the particle (see Jeffery 1922). However if the Reynolds number (Re) is small but non-zero so that fluid inertia has a small but not a negligible effect, then one would expect the particle motion to be not quite periodic, there being a slow change in the value of the orbit constant C. This drift in orbit constant may be determined by modifying the general theory of §§ 3 to 6 and by making now a double expansion in terms of the two parameters Re and κ.

The general result given by equations (6.2) and (6.3) for the force per unit length acting on a long slender body gives a value of zero for the case in which $(\mathbf{U}-\mathbf{U}^*)$ is identically zero at all points on the body centre-line. For such a case it is therefore necessary to take the expansion to a higher order in κ. However, this is a very important case and will be discussed in a future paper since in order to determine the equivalent axis ratio r_e of a long slender body for its motion in shear flow (see Jeffery 1922), it is necessary to consider a long slender body at rest with axis in the r_1 direction (see figure 4) placed in a shear flow given by

$$\mathbf{U} = (r_2, 0, 0),$$

and since $\mathbf{U}^* = 0$, one has for this case,

$$\mathbf{U}-\mathbf{U}^* = 0$$

at all points on the centre-line $r_2 = r_3 = 0$.

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